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NONLINEAR AND BUCKLING ANALYSIS IN PLANAR CURVED BARS

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Abstract—The decoupling of the nonlinear system governing the equilibrium of a deformed planar curved bar subjected to a general compressive loading function is presented. Also, the solution of the differential equation with variable coefficients with respect to the radial displacement of a circular bar, subjected to terminal concentrated forces, was achieved in form of hypergeometric functions; the limit of this solution led to the exact expression of the radial displacement of a flexible shallow arch when the axial force was assumed constant. Finally, the buckling analysis of several shallow or not arches under terminal concentrated forces or uniformly distributed pressure was investigated and the buckling loads were given in a closed form.

1. INTRODUCTION

Buckling of shallow arches and rings under uniform external pressure is a well-studied problem. Timoshenco and Gere[1], Boresi[2], Wempner and Kesti[3] have examined this problem, neglecting the effect of compression on bending of the ring. Also, in order to take into account the effect of a uniform pressure on the deformation of a circular arch or ring, they assumed only inextensional deformations and considered that the internal forces of any cross section reduce to a constant axial force and a bending moment.

Smith and Simitses[4] examined the three loading cases of circular rings and arches in the derivation of the critical loading including the effect of transverse shear, i.e. the case when the force remains normal to the deformed ring during buckling; the case when the force remains parallel to its initial direction during buckling, and the case when the force remains directed toward the initial center of curvature during buckling (centrally directed pressure). It must be noticed here that the authors of [1–3] presented a derivation only for the third case loading.

On the other hand, Fung and Kaplan[5] as well as Gjelsvik and Bodner[6] had derived approximate solutions for various shallow arch buckling problems. All these investigations are valid for arches with zero initial thrust.

Finally, Batterman[9, 10] showed that a direct and consistent scheme for solving the ring buckling problems, as well as all curved buckling problems of any material, is provided by the rate-equation approach.

In the present investigation the mathematical formulation and the solution technique of the buckling problem of a planar bar subjected to a general compressive external loading are presented. Taking into consideration that the curvature of the bar remains unchanged after deformation, the previous problem consists of a system of six coupled nonlinear differential equations with respect to the generalized forces and displacements of the center-line. In the three first equations of this system (equilibrium equations) the effect of all generalized displacements has been taken into account (theory of the second order); a fact that is valid in the well-known Euler's buckling problem of a straight bar. In the following, through appropriate treatments and the results developed in Ref.[11, 12], the previous system was decoupled, so that a differential equation of the fourth order with variable coefficients emerged with respect to the tangential displacement; it must be noticed here that in the foregoing procedure the effects of the axial force and transverse shear have been taken into account.

Moreover, the determination of the radial displacement in form of hypergeometric functions in the case when a circular bar is subjected to a terminal concentrated force was obtained; limiting these expression for the radial displacement, an exact solution of a differential equation with constant coefficients was obtained. The last equation expressed the problem of a shallow flexible bar subjected to the same loading, whose axial force is assumed constant. Then, the buckling of several shallow or deep arches under either terminal concentrated forces or a uniformly distributed pressure was investigated and the critical buckling load was given in a closed form.

2. MATHEMATICAL FORMULATION

We consider a planar curved bar of uniform cross section and curvature $\kappa(s) = 1/R(s)$, subjected to a co-planar compressive general loading; here s is the arc length. We consider also that after deformation the curvature function remains unchanged. We assume the principal triad with unit vectors $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ referred to a generic point Ω of the center line of the bar, so that: $\mathbf{t} = (t_1(s), t_2(s), 0)$ is the unit tangent vector pointing to the direction of s increasing, $\mathbf{n} = (n_1(s), n_2(s), 0)$ is the unit normal vector pointing to the center of curvature and $\mathbf{b} = (0, 0, 1)$ is the unit constant binormal vector; $\mathbf{b}$ is defined in such a way that the principal triad to be a right-handed system. It must be noticed also that the axes of $\mathbf{n}$ and $\mathbf{b}$ coincide with the principal axes of inertia of the cross section. Finally, we denote $+\varphi$ the angle subtended by the positive x-axis and the positive t-axis (Fig. 1a). Based on the previous symbolisms it is valid that:

$$ds = -\,d\varphi/\kappa(s). \tag{2.1}$$

Also, if $x = x(s)$, $y = y(s)$ are the analytical equations of the center-line of the bar, the well-known Serret–Frenet formulae for the point Ω may be written as:

$$\begin{aligned} t_1 &= x',\ t_2 = y' \\ n_1 &= x''/\kappa,\ n_2 = y''/\kappa \end{aligned} \tag{2.2a}$$

or, based on relation (2.1), we have:

$$\begin{aligned} t_1 &= -\,\kappa\dot{x},\ t_2 = -\,\kappa\dot{y} \\ n_1 &= -\,\dot{t}_1,\ n_2 = -\,\dot{t}_2 \end{aligned} \tag{2.2b}$$

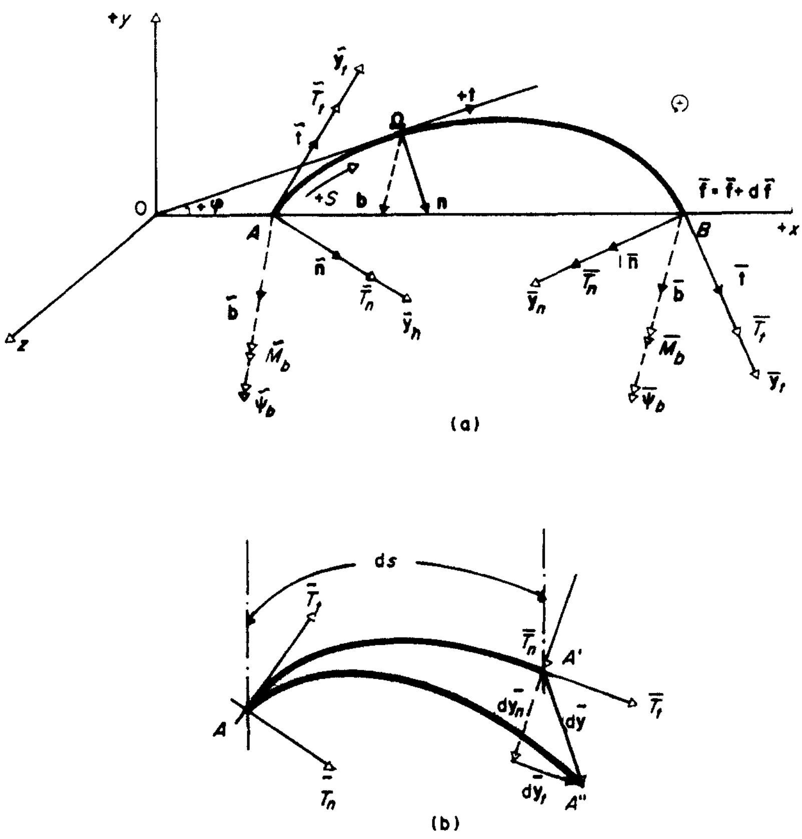

Fig. 1. Geometry and sign convention of a planar curved bar (1a); free body diagram (1b).

where primes designate differentiations with respect to s, dots differentiations with respect to φ and:

$$t_1 = \cos\varphi,\ t_2 = \sin\varphi,\ n_1 = \sin\varphi,\ n_2 = -\cos\varphi.$$

Based on Refs.[11, 12] the nonlinear equilibrium differential equations (in terms of generalized forces and displacements) of an arc element with respect to the principal triad are given by:

$$\left.\begin{array}{ll} T_t' = \kappa T_n + q_t & \text{(2.3a)} \\ T_n' = -\kappa T_t + q_n & \text{(2.3b)} \\ M_b' = -T_n + m_b + T_t y_n' - T_n y_t' & \text{(2.3c)} \end{array}\right\} \text{(2.3)}$$

$$\left.\begin{array}{ll} \psi_b' = -\alpha_3 M_b & \text{(2.4a)} \\ y_t' = \kappa y_n - \beta_1 T_t & \text{(2.4b)} \\ y_n' = -\kappa y_t + \psi_b - \beta_2 T_n & \text{(2.4c)} \end{array}\right\} \text{(2.4)}$$

where $T = (T_t, T_n, 0)$; $M = (M_t, M_n, 0)$ represent the vectors of internal forces and moments; $Y = (y_t, y_n, 0)$, $\Psi = (0, 0, \psi_b)$ the vectors of translations and rotations and $Q = (q_t, q_n, 0)$; $M = (0, 0, m_b)$ the vectors of external forces and moments respectively. Also, β_1, β_2, α_3 are constants given by:

$$\beta_1 = 1/EF,\ \beta_2 = 1/G\nu F,\ \alpha_3 = 1/EI_b \tag{2.5}$$

where E,G are the moduli of elsaticity and shear respectively; F is the area of the cross section; I_b represents the principal moment of inertia of the cross section about the b axis and ν is a coefficient depending on the shape of the cross section. The nonlinear term $T_t y_n' - T_n y_t'$ denotes the additional moment $T_t\,\mathrm{d}y_n - T_n\,\mathrm{d}y_t$, which is caused by the axial force T_t and the transverse shear T_n (Fig. 1b).

It must be noticed here that the extension of the center-line of the bar is given by eqn (2.4b), which can equivalently written as:

$$\varepsilon = \kappa y_n - y_t' \tag{2.6}$$

where ε is the axial strain.
Relation (2.6) coincides with the relation given in ([1], p. 282).

3. DECOUPLING AND SOLUTION OF SYSTEM (2.3)–(2.4)

Equations (2.3a) and (2.3b), after differentiation and successive substitutions, lead to:

$$\begin{gathered} (T_t'/\kappa)' + \kappa T_t = q_n + (q_t/\kappa)' \\ T_n' = -\kappa T_t + q_n. \end{gathered} \tag{3.1}$$

The first of (3.1) is a self-adjoint differential equation whose general solution has been given in Refs.[11, 12] in a closed form as follows:

$$T_t = C_1 \cos\varphi + C_2 \sin\varphi + f \tag{3.2}$$

where f is a particular integral under the form:

$$f = \cos\varphi \int_0^s \sin\varphi \frac{q_n + (q_t/\kappa)'}{\kappa}\,\mathrm{d}s - \sin\varphi \int_0^s \cos\varphi \frac{q_n + (q_t/\kappa)'}{\kappa}\,\mathrm{d}s$$

and C_1, C_2 are the constants of integration, which can be determined through suitable boundary conditions. Thus, T_t and T_n can be considered as known functions.

From eqn (2.4b) and (2.4c) it can be readily derived:

$$(y_t'/\kappa)' + \kappa y_t = \psi_b + g \tag{3.3}$$

where

$$g = -\beta_1(T_t/\kappa)' - \beta_2 T_n.$$

Based on relations (3.3), eqns (2.4a), (2.4b), (2.3c), after differentiations and substitutions, become:

$$\begin{aligned} \psi_b &= (y_t'/\kappa)' + \kappa y_t - g \\ y_n &= y_t'/\kappa + \beta_1 T_t/\kappa \\ M_b &= -1/\alpha_3[(y_t'/\kappa)'' + (\kappa y_t)' - g']. \end{aligned} \tag{3.4}$$

Finally, after differentiation of eqn (2.4a) and based on (2.3c), (3.4) the following complete linear differential equation of the fourth order with variable coefficients is obtained:

$$(y_t'/\kappa)''' + (\kappa y_t)'' + \alpha_3 T_t(y_t'/\kappa)' - \alpha_3 T_n y_t' = z \tag{3.5}$$

where

$$z = -\beta_1(T_t/\kappa)''' - \beta_2 T_n'' + \alpha_3[T_n - m_b - \beta_1 T_t(T_t/\kappa)']$$

and y_t is the tangential displacement.

So, the problem of the deformation of a planar curved bar subjected to a general compressive co-planar external loading leads to the differential equation (3.5) with respect to the tangential displacement. In the formulation of this equation the effects of the axial force T_t and transverse shear T_n have been taken into account. Also, the effect of T_n is introduced by the term $\alpha_3 T_n y_t'$; taking into consideration that this effect can be neglected, eqn (3.5) can take the final form:

$$(y_t'/\kappa)''' + (\kappa y_t)'' + \alpha_3 T_t(y_t'/\kappa)' = z. \tag{3.6}$$

The general integral of (3.6) can be solved only numerically (i.e. through Runge–Kutta's method) for suitable boundary conditions.

The aim of this investigation is to provide the closed form solution of eqn (3.6) in the case of a circular bar and afterwards to determine the buckling load of this bar for several cases of loading and response. Thus, if $\kappa(s) = 1/\mathbf{R} = \text{const}$, eqn (3.6) based on relations (2.4b) and (2.1) can be transformed to:

$$y_n''' + (\kappa^2 + \alpha_3 T_t)y_n' = \kappa\beta_1 T_t' + \alpha_3(T_n - m_b) - \beta_2 T_n'' \tag{3.7}$$

where y_n is the radial displacement.

We notice here that, neglecting the term κ^2 in the left-hand member of eqn (3.7) and using relation (2.6), the resulting expression coincides with the one given in eqn (6) of Ref [8].

Transforming relation (3.7) through the relation (2.1) the following differential equation results:

$$\dddot{y}_n + \left(1 + \frac{\alpha_3 T_t}{\kappa^2}\right)\dot{y}_n = \frac{\beta_1 \dot{T}_t}{\kappa} - \frac{\alpha_3}{\kappa^3}(T_n - m_b) + \frac{\beta_2 \ddot{T}_n}{\kappa} \tag{3.8}$$

which by the substitution:

$$\dot{y} = h$$

leads to:

$$\ddot{h} + \left(1 + \frac{\alpha_3 T_t}{\kappa^2}\right)h = \frac{\beta_1 \dot{T}_t}{\kappa} - \frac{\alpha_3}{\kappa^3}(T_n - m_b) + \frac{\beta_2 \ddot{T}_n}{\kappa}. \tag{3.9}$$

Let us consider now a circular bar **AB** compressed by two forces **P** acting along its cord (Fig. 2). Then, it is valid:

$$q_t = q_n = m_b = 0$$
$$T_t = \mathbf{P}\cos\varphi,\ T_n = \mathbf{P}\sin\varphi \tag{3.10}$$

Introducing now relations (3.10) in (3.9) the following Mathieu's equation arises:

$$\ddot{h} + (1 + \lambda^2\cos\varphi)h = -\frac{\mathbf{P}\mu}{\kappa}\sin\varphi \tag{3.11}$$

where:

$$\lambda^2 = \alpha_3\mathbf{P}/\kappa^2,\ \mu = \beta_1 + \beta_2 + (\alpha_3/\kappa^2). \tag{3.12}$$

Accepting that for arches with epicentral angle $0 \leqq \varphi \leqq \pi/2$ it is valid that:

$$\cos\varphi \cong 1 - \varphi^2/2$$

the homogeneous differential equation of (3.11) takes the form:

$$\ddot{h} + (1 + \lambda^2 - \lambda^2\varphi^2/2)h = 0. \tag{3.13}$$

The last equation through the substitution:

$$\frac{\lambda 2^{1/2}}{2}\varphi^2 = p$$

has a solution, see[13], the function:

$$h = \left(\frac{\lambda 2^{1/2}}{2}\right)^{1/4} p^{-1/4}[C_1 M_{k,m}(p) + C_2 M_{k,-m}(p)] \tag{3.14}$$

where:

$$\begin{aligned} M_{k,m}(p) &= p^{3/4}\exp(-p/2)_1F_1(\alpha,\gamma;p) \\ M_{k,-m}(p) &= p^{1/4}\exp(-p/2)_1F_1(\bar{\alpha},\bar{\gamma};p) \\ k &= (1+\lambda^2)2^{1/2}/4\lambda,\ m = 1/4 \\ \alpha &= [3\lambda - (1+\lambda^2)2^{1/2}]/4\lambda,\ \gamma = 3/2, \\ \bar{\alpha} &= [\lambda - (1+\lambda^2)2^{1/2}]/4\lambda,\ \bar{\gamma} = 1/2 \end{aligned} \tag{3.15}$$

and C_1, C_2 are integration constants.

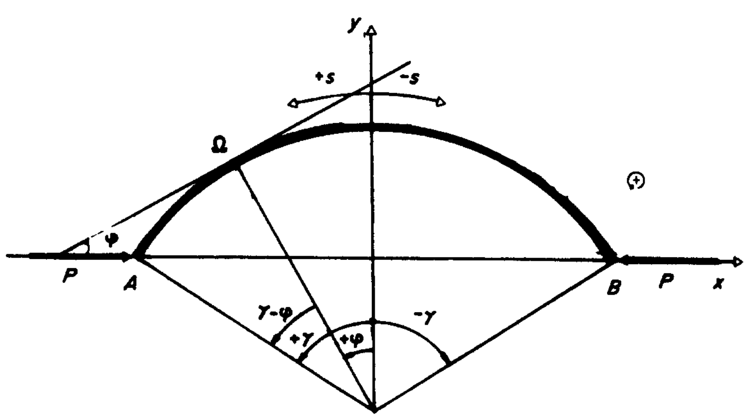

Fig. 2. Geometry and sign convention of a circular bar.

In relations (3.15) ${}_1\mathbf{F}_1$ is the degenerate hypergeometric function given by:

$$ {}_1\mathbf{F}_1(\alpha, \gamma: p) = 1 + \frac{\alpha}{\gamma}\frac{p}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{p^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{p^3}{3!} + \cdots $$

So, the two linearly independent particular solutions of eqn (3.13) are:

$$ h_1(p) = p^{1/2}\exp(-p/2)\,{}_1\mathbf{F}_1(\alpha, \gamma; p) $$
$$ h_2(p) = \exp(-p/2)\,{}_1\mathbf{F}_1(\bar{\alpha}, \bar{\gamma}; p) $$

which can be equivalently written:

$$ h_1(p) = \sum_{n=0}^{\infty} A_n \exp(-p/2) p^{n+(1/2)} $$
$$ h_2(p) = \sum_{n=0}^{\infty} \bar{A}_n \exp(-p/2) p^n \tag{3.16} $$

where:

$$ A_0 = 1, A_1 = \alpha/\gamma, A_2 = \alpha(\alpha+1)/\gamma(\gamma+1)2!, \ldots $$
$$ \bar{A}_0 = 1, \bar{A}_1 = \bar{\alpha}/\bar{\gamma}, \bar{A}_2 = \bar{\alpha}(\bar{\alpha}+1)/\bar{\gamma}(\bar{\gamma}+1)2!, \ldots \tag{3.17} $$

Applying now the method of variation of constants, the general integral of (3.11) can be expressed as:

$$ h(p) = C_1 h_1(p) + C_2 h_2(p) + h_2(p)\mathbf{F}_{1,h_1(p)} - h_1(p)\mathbf{F}_{2,h_2(p)} \tag{3.17a} $$

where $\mathbf{F}_{1,f(p)}$ is the real functional:

$$ \mathbf{F}_{1,f(p)} = \rho \int_0^p \frac{\sin \nu p^{1/2}}{h_1(p)\bar{h}_2(p) - \bar{h}_1(p)h_2(p)} f(p)\,\mathrm{d}p \tag{3.17b} $$

and:

$$ \bar{h}_1(p) = \sum_{n=0}^{\infty} A_n \exp(-p/2)\frac{2n+1}{2} p^{(2n-1)/2} $$
$$ \bar{h}_2(p) = \sum_{n=0}^{\infty} \bar{A}_n \exp(-p/2) n p^{n-1} \tag{3.17c} $$
$$ \rho = \frac{\mathbf{P}\mu}{-\kappa}, \; \nu = \left(\frac{2^{1/2}}{\lambda}\right)^{1/2}. $$

Based on the previous procedure and on the relation $\dot{y}_n = h$ the radial displacement y_n can be easily derived; in fact, introducing the incomplete gamma function:

$$ \gamma(a, x) = \int_0^x \exp(-t)t^{a-1}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n x^{a+n}}{n!(a+n)} $$

and taking into account that:

$$ (\lambda 2^{1/2}/2)\varphi^2 = p $$

we have:

$$ y_n(\varphi) = C_1 \sum_{n=0}^{\infty} A_n 2^{(2n+3)/2}\gamma\left(\frac{2n+3}{2}, \frac{\lambda 2^{1/2}}{4}\varphi^2\right) + C_2 \sum_{n=0}^{\infty} \bar{A}_n 2^{n+1}\gamma\left(n+1, \frac{\lambda 2^{1/2}}{4}\varphi^2\right) $$
$$ + \lambda 2^{1/2}\int_0^{\varphi} [h_2(\varphi)\mathbf{F}_{1,h_1(\varphi)} - h_1(\varphi)\mathbf{F}_{1,h_2(\varphi)}]\varphi\,\mathrm{d}\varphi + C_3 \tag{3.18} $$

where C_3 is a new integration constant.

After the determination of y_n the quantities y_t, ψ_b, M_b result through eqns (3.4) and (2.1). It must be noticed here that the expression of the function y_t introduces another integration constant C_4. The constants $C_i (i = 1, \ldots, 4)$, as well as the buckling load for several cases of response and loading of the circular bar can be achieved under suitable boundary conditions.

For large values of the parameter $|k|$, i.e. for shallow and flexible bars, (equivalently for $\lambda^2 \gg 1$), we have:

$$\lim_{|k| \gg 0} M_{k,m} = \frac{1}{\sqrt{\pi}} \Gamma(2m+1) k^{-m-(1/4)} p^{1/4} \cos\left(2\sqrt{(kp)} - m\pi - \frac{1}{4}\pi\right)$$
$$\lim_{|k| \gg 0} M_{k,-m} = \frac{1}{\sqrt{\pi}} \Gamma(-2m+1) k^{m-(1/4)} p^{1/4} \cos\left(2\sqrt{(kp)} + m\pi - \frac{1}{4}\pi\right) \tag{3.19}$$

where Γ is the gamma function.

Because of:

$$k = \frac{(1+\lambda^2)2^{1/2}}{4\lambda}, \pm m = \pm\frac{1}{4}, \Gamma(2m+1) = \frac{\sqrt{\pi}}{2}, \Gamma(-2m+1) = 2\sqrt{\pi}$$

relations (3.19) lead to:

$$\lim_{|k| \gg 0} M_{k,m} = k^{-1/4} p^{1/4} \sin(1+\lambda^2)^{1/2} \varphi$$
$$\lim_{|k| \gg 0} M_{k,-m} = 2p^{1/4} \cos(1+\lambda^2)^{1/2} \varphi. \tag{3.20}$$

So, based on formula (3.14) and relations (3.20) the particular integrals of the differential eqn (3.13) result to:

$$h_1(\varphi) = \sin\sqrt{(1+\lambda^2)}\varphi$$
$$h_2(\varphi) = \cos\sqrt{(1+\lambda^2)}\varphi. \tag{3.21}$$

The solutions (3.21) coincide with those of the homogeneous differential equation (3.11) through the approximation $\cos\varphi \cong 1$, or $\alpha_3 T_t/\kappa^2 = \text{const}$. Consequently, it can be concluded that, for shallow and flexible circular bars compressed by two forces **P** acting along their cord, it is valid:

$$\frac{\alpha_3 T_t}{\kappa^2} = \text{const.}$$

Thus, eqn (3.11) becomes a differential equation with constant coefficients:

$$\ddot{h} + u^2 h = -\frac{\mathbf{P}\mu}{\kappa} \sin\varphi \tag{3.22}$$

where:

$$u^2 = 1 + \lambda^2 = 1 + \frac{\alpha_3 \mathbf{P}}{\kappa^2} = \text{const.} \tag{3.22a}$$

By now, the functions y_n, y_t, ψ_b, M_b, based on relations (3.4), (3.11), (3.12) and (3.21) and on notations of Fig. 2, derive as:

$$y_n = \frac{1}{u}\left[-C_1\cos u\varphi + C_2\sin u\varphi + C_3 + \frac{\mathbf{P}u\mu}{\kappa(u^2-1)}\cos\varphi\right]$$

$$y_t = \frac{1}{u^2}\left[C_1\sin u\varphi + C_2\cos u\varphi - C_3u\varphi + C_4 + \frac{\mathbf{P}u^2}{\kappa}\left(\beta_1 - \frac{u}{u^2-1}\right)\sin\varphi\right]$$

$$\psi_b = -\frac{\kappa}{u^2}\left[C_1(u^2-1)\sin u\varphi + C_2(u^2-1)\cos u\varphi + C_3u\varphi - C_4 - \frac{(\beta_1+\beta_2)\mathbf{P}u^2}{\kappa}\sin\varphi\right] \quad (3.23)$$

$$M_b = -\frac{\kappa^2}{u\alpha_3}\left[C_1(u^2-1)\cos u\varphi - C_2(u^2-1)\sin u\varphi + C_3 - \frac{(\beta_1+\beta_2)\mathbf{P}u}{\kappa}\cos\varphi\right]$$

where the dimensionless quantity u is given by the relation (3.22a). In the following, based on eqns (3.22)–(3.23) and through the suitable boundary conditions of the problem, the critical buckling loadings of several cases of shallow or not circular bars will be determined. This procedure is based on the research of limit points which are found through the solution of a transcendental equation corresponding to each of the previous cases.

4. BUCKLING OF SHALLOW CIRCULAR BARS

4.1 *The cantilever circular bar*

Consider the circular cantilever bar **AB**, with the one end **A** fully fixed, subjected to two compressive forces acting along its cord (Fig. 3a). The boundary conditions for this problem

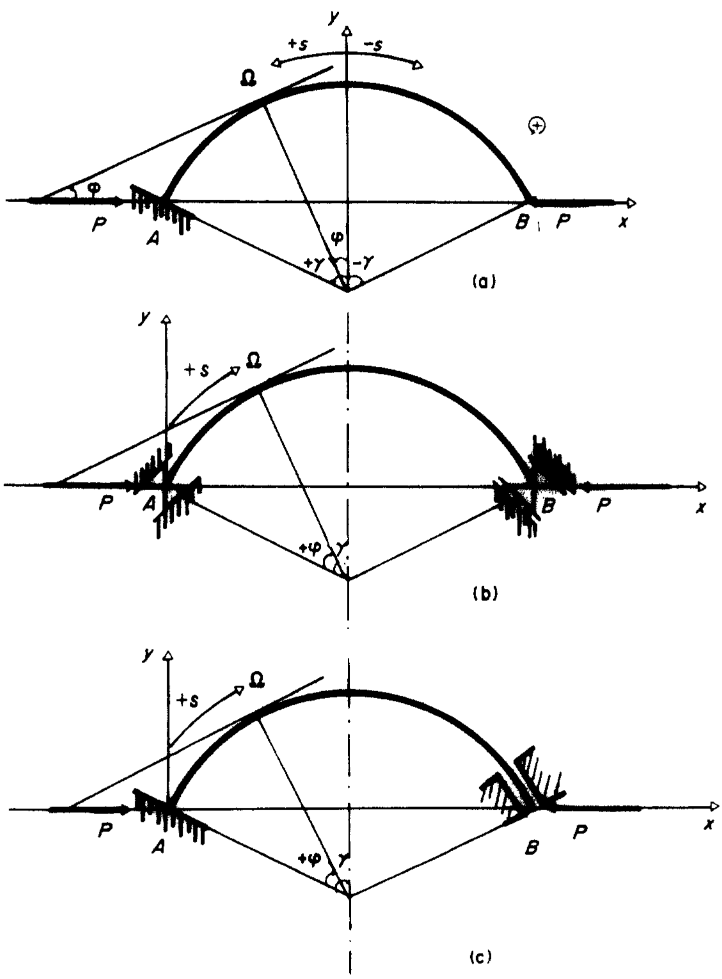

Fig. 3. A cantilever circular bar subjected to terminal forces (3a); a circular bar with hinged ends subjected to terminal forces (3b); a circular bar with both fixed ends subjected to terminal forces (3c).

are:

$$
\begin{aligned}
y_n(\gamma) &= 0 \\
y_t(\gamma) &= 0 \\
\psi_b(\gamma) &= 0 \\
M_b(\gamma) &= 0 \\
M_b(-\gamma) &= 0.
\end{aligned}
\tag{4.1}
$$

Taking into account eqns (3.23) the four last of the previous conditions lead, after algebraic manipulations, to the following relations:

$$
\begin{aligned}
C_1 &= \frac{\mathbf{P}}{\kappa}\frac{\sin\gamma}{\sin u\gamma}\left(\beta_2 + \frac{u}{u^2-1}\right) \\
C_2 &= 0 \\
C_3 &= -\frac{\mathbf{P}}{\kappa}\{-\cot u\gamma \sin\gamma[\beta_2(u^2-1)+\mu] + u(\beta_1+\beta_2)\cos\gamma\} \\
C_4 &= \frac{\mathbf{P}}{\kappa}\{-u\gamma\cot u\gamma \sin\gamma[\beta_2(u^2-1)+\mu] + (\beta_1+\beta_2)(u^2\gamma\cos\gamma - \sin\gamma)\}
\end{aligned}
\tag{4.2}
$$

where:

$$u = (1 + \alpha_3\mathbf{P}/\kappa^2)^{1/2}.$$

Moreover, the first of (4.1), based on eqn (4.2), leads to the following transcendental equation with respect to u:

$$u \tan\gamma \cot u\gamma = 1 + \frac{\beta_1}{\dfrac{\beta_1+\beta_2+(\alpha_3/\kappa^2)}{u^2-1} + \beta_2}. \tag{4.3}$$

This equation, determining the critical buckling load $\mathbf{P}_{cr}$, can be, furthermore, simplified as follows:

We notice that:

$$\frac{\alpha_3}{\kappa^2} \gg \beta_1 + \beta_2.$$

In fact, for a bar of large radius of curvature and of quandragular cross section of depth $\mathbf{h}(\nu = 5/6)$ and because of relations (2.5) the last inequality leads in the following expression (for $G = E/2$):

$$\mathbf{R}^2 \gg \mathbf{h}^2/3{,}53$$

which is obviously valid because $\mathbf{R}^2 \gg \mathbf{h}^2$. So, $\beta_1 = \beta_2 \cong 0$. Also, as $\beta_1\mathbf{P} \ll 1$ the right member of (4.3) becomes:

$$1 + \frac{\beta_1}{\dfrac{\beta_1+\beta_2+(\alpha_3/\kappa^2)}{u^2-1} + \beta_2} \cong 1 + \beta_1\mathbf{P} \cong 1.$$

Consequently the transcendental eqn (4.3) takes the form:

$$u \tan\gamma \cot u\gamma = 1. \tag{4.4}$$

If $u_1 (u_1 > 1)$ is the smallest root of (4.4), the buckling load $\mathbf{P}_{cr}$ results:

$$\mathbf{P}_{cr} = (u_1{}^2 - 1)\kappa^2 EI_b. \tag{4.5}$$

Table 1. Dimensionless ratio $(1+\alpha_3 \mathbf{P}_{cr}/\kappa^2)^{1/2}$

γ	10°	15°	30°	45°	60°	75°	90°
u_1	25.755	17.185	8.621	5.782	4.375	3.544	3

In Table 1 several values of $u_1 = (1+\alpha_3 \mathbf{P}_{cr}/\kappa^2)^{1/2}$ are given for various values of the angle γ.

We notice here that the relations (4.4), (4.5) and some of the results of the Table 1 coincide with those given in Ref. ([1], p. 300), where the buckling of a uniformly compressed arch with both fixed ends is examined.

4.2 *The bar with hinged ends*

If a circular bar with hinged ends is compressed again by two forces **P** acting along its cord, the boundary conditions can be expressed as:

$$\begin{gathered} y_n(0) = y_t(0) = y_n(2\gamma) = y_t(2\gamma) = 0 \\ M_b(0) = M_b(2\gamma) = 0. \end{gathered} \tag{4.6}$$

Taking into account that it is valid (Fig. 3b):

$$T_t = \mathbf{P}\cos(\gamma-\varphi),\ T_n = \mathbf{P}\sin(\gamma-\varphi),\ \mathrm{d}s = \mathbf{R}\,\mathrm{d}\varphi$$

eqns (3.23) take the form:

$$\begin{aligned} y_n &= \frac{1}{u}\left[-C_1\cos u\varphi + C_2\sin u\varphi + C_3 + \frac{u\mu\mathbf{P}}{\kappa(u^2-1)}\cos(\gamma-\varphi)\right] \\ y_t &= \frac{1}{u^2}\left[-C_1\sin u\varphi - C_2\cos u\varphi + C_3u\varphi + C_4 + \frac{\mathbf{P}u^2}{\kappa}\left(\beta_1 - \frac{\mu}{u^2-1}\right)\sin(\gamma-\varphi)\right] \\ \psi_b &= \frac{\kappa}{u^2}\Big[C_1(u^2-1)\sin u\varphi + C_2(u^2-1)\cos u\varphi + C_3u\varphi + C_4 \\ &\quad + \frac{\mathbf{P}}{\kappa}u^2(\beta_1+\beta_2)\sin(\gamma-\varphi)\Big] \\ M_b &= \frac{\kappa^2}{\alpha_3 u}\left[-C_1(u^2-1)\cos u\varphi + C_2(u^2-1)\sin u\varphi - C_3 + \frac{u\mathbf{P}}{\kappa}(\beta_1+\beta_2)\cos(\gamma-\varphi)\right]. \end{aligned} \tag{4.7}$$

The combination of (4.7) and (4.6) leads to the following algebraic system:

$$-C_1 + C_3 + \frac{u\mu\mathbf{P}}{\kappa(u^2-1)}\cos\gamma = 0 \tag{4.8a}$$

$$-C_1\cos 2u\gamma + C_2\sin 2u\gamma + C_3 + \frac{u\mu\mathbf{P}}{\kappa(u^2-1)}\cos\gamma = 0 \tag{4.8b}$$

$$-C_2 + C_4 + \frac{u^2\mathbf{P}}{\kappa}\left(\beta_1 - \frac{\mu}{u^2-1}\right)\sin\gamma = 0 \tag{4.8c}$$

$$-C_1\sin 2u\gamma - C_2\cos 2u\gamma + C_3 2u\gamma + C_4 - \frac{u^2\mathbf{P}}{\kappa}\left(\beta_1 - \frac{\mu}{u^2-1}\right)\sin\gamma = 0 \tag{4.8d}$$

$$-C_1(u^2-1) - C_3 + \frac{u\mathbf{P}(\beta_1+\beta_2)}{\kappa}\cos\gamma = 0 \tag{4.8e}$$

$$-C_1(u^2-1)\cos 2u\gamma + C_2(u^2-1)\sin 2u\gamma - C_3 + \frac{u\mathbf{P}}{\kappa}(\beta_1+\beta_2)\cos\gamma = 0 \tag{4.8f}$$

from the decoupling of which we have:

$$
\begin{aligned}
C_1 &= \frac{\mathbf{P}\cos\gamma}{u\kappa}\left(\beta_1+\beta_2+\frac{\mu}{u^2-1}\right)\\
C_2 &= -\frac{\mathbf{P}\cos\gamma}{u\kappa}\left(\beta_1+\beta_2+\frac{\mu}{u^2-1}\right)\tan u\gamma\\
C_3 &= \frac{\mathbf{P}\cos\gamma}{u\kappa}(\beta_1+\beta_2-\mu)\\
C_4 &= -\frac{\mathbf{P}\cos\gamma}{u\kappa}\left[\left(\beta_1+\beta_2+\frac{\mu}{u^2-1}\right)\tan u\gamma+u^3\left(\beta_1-\frac{\mu}{u^2-1}\right)\tan\gamma\right].
\end{aligned}
\tag{4.9}
$$

The addition of eqns (4.8b) and (4.8f) becomes an identity; therefore, from the six boundary conditions (4.6) only the five are independent. Finally, substituting the coefficients C_i in eqn (4.8d) the following transcendental equation derives:

$$
\frac{\beta_1+\beta_2+\frac{\mu}{u^2-1}}{\beta_1+\beta_2-\mu}\,\frac{1}{u\gamma}\tan u\gamma+\frac{\beta_1-\frac{\mu}{u^2-1}}{\beta_1+\beta_2-\mu}\,\frac{u^2}{\gamma}\tan\gamma=1. \tag{4.10}
$$

By putting $\beta_1=\beta_2\cong 0$, eqn (4.10) becomes:

$$
u\cot u\gamma[u^2(\tan\gamma-\gamma)+\gamma]=1. \tag{4.11}
$$

If $u_1(u_1>1)$ is the smallest root of the last equation the buckling load $\mathbf{P}_{cr}$ can be derived from the formula (4.5). So, we formulate the Table 2. Here we remark that the critical load is greater than the corresponding load given in Table 1, i.e. the buckling load of a circular bar with hinged ends is greater than that of a cantilever one; this result is in agreement with the result given by Euler for the buckling of straight rods.

4.3 *The bar with both fixed ends*

Here the boundary conditions are, (Fig. 3c):

$$
\begin{aligned}
&y_n(0)=y_t(0)=y_n(2\gamma)=y_t(2\gamma)=0\\
&\psi_b(0)=\psi_b(2\gamma)=0.
\end{aligned}
\tag{4.12}
$$

Substituting eqns (4.12) into (4.7) and, after the decoupling of the result system, we obtain:

$$
\begin{aligned}
C_1 &= \frac{\mathbf{P}}{\kappa}\left(\beta_2+\frac{\mu}{u^2-1}\right)\sin\gamma\cot u\gamma\\
C_2 &= -\frac{\mathbf{P}}{\kappa}\left(\beta_2+\frac{\mu}{u^2-1}\right)\sin\gamma\\
C_3 &= \frac{\mathbf{P}}{\kappa}\left[\left(\beta_2+\frac{\mu}{u^2-1}\right)\cot u\gamma-\frac{u\mu}{u^2-1}\cot\gamma\right]\sin\gamma\\
C_4 &= -\frac{\mathbf{P}}{\kappa}(\beta_1u^2+\beta_2-\mu)\sin\gamma
\end{aligned}
\tag{4.13}
$$

as well as, an analogous to (4.10) transcendental equation, from which the critical load can be determined:

$$
u\tan\gamma\cot u\gamma=\frac{u^2\mu}{\left(\beta_2+\frac{\mu}{u^2-1}\right)(u^2-1)}+\frac{\beta_1u^2+\beta_2-\mu}{\left(\beta_2+\frac{\mu}{u^2-1}\right)\gamma}\tan\gamma. \tag{4.14}
$$

Table 2. Dimensionless ratio $(1+\alpha_3\mathbf{P}_{cr}/\kappa^2)^{1/2}$

γ	10°	15°	30°	45°	60°	75°	90°
u_1	26,866	17,897	14,990	9,994	7,475	5,993	5,016

Table 3. Dimensionless ratio $(1+\alpha_3 P_{cr}/\kappa^2)^{1/2}$

γ	10°	15°	30°	45°	60°	75°	90°
u_1	32,948	21,994	17,379	11,598	8,709	6,979	5,826

Assuming $\beta_1 = \beta_2 \cong 0$ we have:

$$u \tan \gamma \cot u\gamma = u^2 - \frac{u^2 - 1}{\gamma} \tan \gamma. \tag{4.15}$$

If $u_1 (u_1 > 0)$ is the smallest root of (4.15) we form the Table 3. We notice here that the critical buckling load $\mathbf{P}_{cr}$ is greater than the corresponding $\mathbf{P}_{cr}$ of cases 4.1 and 4.2, as exactly it is valid in Euler's theory of buckling for straight rods.

5. BUCKLING OF A UNIFORMLY COMPRESSED CIRCULAR BAR

5.1 *The bar with hinged ends*

We consider now a circualr bar **AB** with hinged ends, submitted to the action of a uniformly distributed pressure q, (Fig. 4a). Then:

$$\begin{aligned} &T_t = T_{t_A} = q\mathbf{R} = \text{const.} \\ &T_n = 0 \\ &M_b = 0 \text{ before deformation.} \end{aligned} \tag{5.1}$$

We remark that the axial force T_t is constant for every point of the center line; so, since the second member of eqn (3.9) does not exist, the differential equation (3.8) has the following general integral:

$$-C_1 \frac{1}{u} \cos u\varphi + C_2 \frac{1}{u} \sin u\varphi + C_3.$$

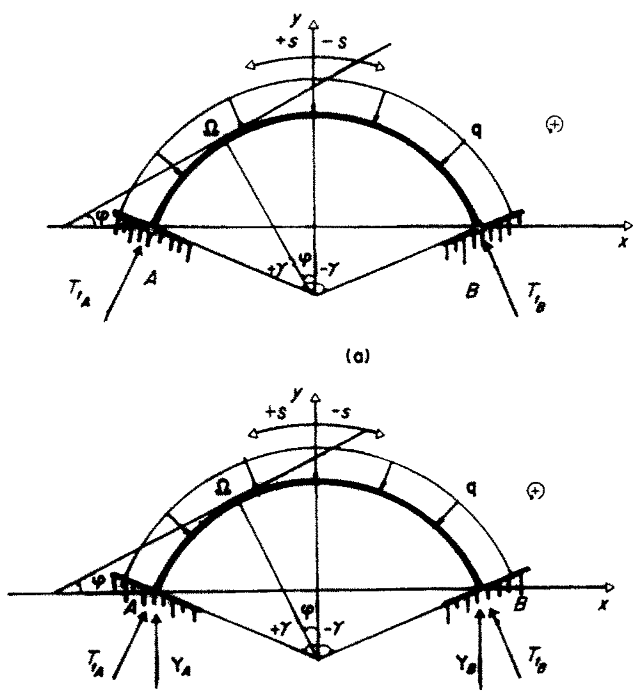

Fig. 4. A circular bar with hinged ends subjected to a uniformly distributed pressure (4a); a circular bar with both fixed ends subjected to a uniformly distributed pressure (4b).

Now, based on the previous procedure and on relations (3.4) the functions y_n, y_t, ψ_b, M_b can be derived as follows:

$$y_n = \frac{1}{u}(-C_1\cos u\varphi + C_2\sin u\varphi + C_3)$$
$$y_t = \frac{1}{u^2}(-C_1\sin u\varphi - C_2\cos u\varphi + C_3u\varphi + C_4 - u^2\beta_1 q\mathbf{R}^2\varphi)$$
$$\psi_b = \frac{1}{u^2\mathbf{R}}[C_1(u^2-1)\sin u\varphi + C_2(u^2-1)\cos u\varphi + C_3u\varphi + C_4 - u^2\beta_1 q\mathbf{R}^2\varphi]$$
$$M_b = \frac{1}{u\alpha_3\mathbf{R}^2}[-C_1(u^2-1)\cos u\varphi + C_2(u^2-1)\sin u\varphi - C_3 + u\beta_1 q\mathbf{R}^2].$$

Also, because of relations (5.2) the boundary conditions given by eqns (4.6) lead to the following algebraic system:

$$-C_1 + C_3 = 0 \tag{5.3a}$$
$$-C_1\cos 2u\gamma + C_2\sin 2u\gamma + C_3 = 0 \tag{5.3b}$$
$$-C_2 + C_4 = 0 \tag{5.3c}$$
$$-C_1\sin 2u\gamma - C_2\cos 2u\gamma + C_3 2u\gamma + C_4 - 2u^2\gamma\beta_1 q\mathbf{R}^2 = 0 \tag{5.3d}$$
$$-C_1(u^2-1) - C_3 + u\beta_1 q\mathbf{R}^2 = 0 \tag{5.3e}$$
$$-C_1(u^2-1)\cos 2u\gamma + C_2(u^2-1)\sin 2u\gamma - C_3 + u\beta_1 q\mathbf{R}^2 = 0. \tag{5.3f}$$

From the combination of (5.3a), (5.3e) and (5.3b), (5.3f) we have:

$$C_1 = C_3 = \beta_1 q\mathbf{R}^2/u$$
$$C_2 = C_4 = -\beta_1 q\mathbf{R}^2 \tan u\gamma/u.$$

We notice here that eqn (5.3c) is an identity; so, from the six boundary conditions (4.6) only the five are independent. Also, eqn (5.3d) becomes:

$$\tan u\gamma = u\gamma(1-u^2). \tag{5.4}$$

If $u_1(u_1 > 1)$ is the smallest root of eqn (5.4) the critical buckling load q_{cr} results from the following relation:

$$q_{cr} = (u_1{}^2 - 1)\frac{EI_b}{\mathbf{R}^3}. \tag{5.5}$$

For various values of the epicentral angle γ the Table 4 is constituted. In the values of the parameter u_1 of the Table 4 the effect of compression of the bar have been taken into account. If this effect is omitted ($\beta_1 = 0$) then we have:

$$\sin 2u\gamma = 0$$
$$\cos 2u\gamma = 0. \tag{5.6}$$

Equations (5.6) are verified simultaneously for:

$$\sin 2u\gamma = 2\pi \Rightarrow u = \pi/\gamma. \tag{5.7}$$

Table 4. Dimensionless ratio $(1 + q_{cr}\mathbf{R}^3/EI_b)^{1/2}$

γ	10°	15°	30°	45°	60°	75°	90°
u_1	27,00167	18,00251	9,00506	6,00771	4,51045	3,61339	3,01661

So, the functions y_n, y_t, ψ_b, M_b are:

$$\begin{aligned} y_n &= \frac{1}{u} C_2 \sin u\varphi \\ y_t &= \frac{1}{u^2} C_2(1 - \cos u\varphi) \\ \psi_b &= \frac{1}{u^2\mathbf{R}} C_2[1 + (u^2 - 1)\cos u\varphi] \\ M_b &= \frac{1}{\alpha_3 u \mathbf{R}^2} C_2(u^2 - 1)\sin u\varphi \end{aligned} \tag{5.8}$$

and the corresponding critical load q_{cr} results:

$$q_{cr} = \frac{EI_b}{\mathbf{R}^2}\left(\frac{\pi^2}{\gamma^2} - 1\right). \tag{5.9}$$

The last formula coincides with the relation given in Ref. ([1], p. 297). Also, the first of (5.8) for $q = q_{cr}$ gives:

$$u y_n = C_2 \sin \frac{\pi}{\gamma} \varphi$$

and for $\varphi = \gamma$

$$u y_n = 0 \Rightarrow y_n = 0.$$

So, it was assumed that the buckled bar had an inflection point at the middle.

Finally, for the determination of the integration constant C_2 of eqn (5.8) the theorem of elastic energy is used, i.e:

$$\int_0^{2\gamma} q \, \mathrm{d}s y_n = \alpha_3 \int_0^{2\gamma} M_b^2 \, \mathrm{d}s$$

from which

$$C_2 = \frac{\alpha_3 q u^2 \mathbf{R}^4}{(u^2 - 1)^2} \frac{1 - \cos 2u\gamma}{u\gamma - \dfrac{\sin 2u\gamma}{4}}. \tag{5.10}$$

Based now on eqn (5.7) we make the Table 5. From the comparison of results in Table 4 and 5 we noticed that, if $\beta_1 = 0$, the buckled load q_{cr} is smaller than the corresponding q_{cr} when $\beta_1 \neq 0$.

5.2 *The bar with both fixed ends*

If the ends of a uniformly compressed bar are fixed, (Fig. 4b), then the following relations are valid before and after deformation respectively:

$$\begin{aligned} T_t &= q\mathbf{R} = \text{const} \\ T_n &= 0 \\ \mathrm{M}_n &= 0 \end{aligned} \tag{5.11a}$$

Table 5. Dimensionless ratio $(1 + q_{cr}\mathbf{R}^3/EI_b)^{1/2}$

γ	10°	15°	30°	45°	60°	75°	90°
u_1	18	12	6	4	3	2,4	2

$$\begin{aligned} T_t &= q\mathbf{R} + y\sin(\gamma-\varphi) \\ T_n &= -y\cos(\gamma-\varphi) \end{aligned} \tag{5.11b}$$

where y is the unknown function of the normal reaction in the fixed ends. Consequently, in this case the general solution of the homogeneous differential equation of (3.8) is:

$$-C_1\frac{1}{u}\cos u\varphi + C_2\frac{1}{u}\sin u\varphi + C_3.$$

Also, the second member of (3.9) becomes:

$$-\mu\mathbf{R}y\cos(\gamma-\varphi)$$

the particular integral of which is:

$$-\frac{\mu\mathbf{R}}{u^2-1}y\cos(\gamma-\varphi).$$

Through a similar procedure to that developed in the previous sections, the following algebraic system results:

$$-C_1 + C_3 + \frac{u\mu\mathbf{R}}{u^2-1}y\sin\gamma = 0 \tag{5.12a}$$

$$-C_1\cos 2u\gamma + C_2\sin 2u\gamma + C_3 - \frac{u\mu\mathbf{R}}{u^2-1}y\sin\gamma = 0 \tag{5.12b}$$

$$-C_2 + C_4 - u^2\mathbf{R}\left(\beta_1 - \frac{\mu}{u^2-1}\right)y\sin\gamma = 0 \tag{5.12c}$$

$$-C_1\sin 2u\gamma - C_2\cos 2u\gamma + C_3 2u\gamma + C_4 - u^2\beta_1 q\mathbf{R}^2 2\gamma - u^2\mathbf{R}\left(\beta_1 - \frac{\mu}{u^2-1}\right)y\cos\gamma = 0 \tag{5.12d}$$

$$C_2(u^2-1) + C_4 - u^2\mathbf{R}(\beta_1+\beta_2)y\cos\gamma = 0 \tag{5.12e}$$

$$C_1(u^2-1)\sin 2u\gamma + C_2(u^2-1)\cos 2u\gamma + C_3 2u\gamma + C_4 - u^2\mathbf{R}(\beta_1+\beta_2)y\cos\gamma - u^2\beta_1 q\mathbf{R}^2 2\gamma = 0 \tag{5.12f}$$

from the decoupling of which we have:

$$\begin{aligned} C_1 &= \mathbf{R}y\cos\gamma\left[-\left(\beta_2+\frac{\mu}{u^2-1}\right)\cot u\gamma + \frac{u\mu}{u^2-1}\frac{\tan\gamma}{\sin^2 u\gamma}\right] \\ C_2 &= \mathbf{R}\left(\beta_2+\frac{\mu}{u^2-1}\right)y\cos\gamma \\ C_3 &= \mathbf{R}y\cos\gamma\left[-\left(\beta_2+\frac{\mu}{u^2-1}\right)\cot u\gamma + \frac{u\mu}{u^2-1}\tan\gamma\cot^2 u\gamma\right] \\ C_4 &= \mathbf{R}(u^2\beta_1+\beta_2-\mu)y\cos\gamma \end{aligned} \tag{5.13}$$

$$C_1u^2\sin 2u\gamma + C_2u^2\cos 2u\gamma - u^2\mathbf{R}\left(\beta_2+\frac{\mu}{u^2-1}\right)y\cos\gamma = 0. \tag{5.14}$$

The combination of eqns (5.13), (5.14) gives:

$$u\tan\gamma\cot u\gamma = \beta_2\frac{u^2-1}{\mu} + 1. \tag{5.15}$$

Putting $\beta_2 \cong 0$ eqn (5.15) becomes:

$$u\tan\gamma\cot u\gamma = 1. \tag{5.16}$$

from which, based on the formula (5.5), the critical buckling load q_{cr} results.

It must be noticed here that the relation (5.16) coincides with (4.4) and so the smallest root of (5.16) takes the same values with those given in Table 1. The unknown function y can be determined through relations (5.13). Indeed, introducing the coefficients C_i in (5.12d) and after sucessive manipulations we have:

$$y(u)=\frac{\beta_1 u^2 q\mathbf{R}\gamma}{\cos\gamma\left[\left(\beta_2+\frac{\mu}{u^2-1}\right)(1-u\gamma\cot u\gamma)+\frac{u\mu\cot u\gamma\tan\gamma}{u^2-1}(u\gamma-1)\right]}. \tag{5.17}$$

If $\beta_2=0$ then:

$$y(u)=\frac{\beta_1 u^2 q\mathbf{R}\gamma(u^2-1)}{\mu\cos\gamma[1-u\gamma\cot u\gamma+u\cot u\gamma\tan\gamma(u\gamma-1)]} \tag{5.17a}$$

and if $\beta_1=0$

$$y(u)=0. \tag{5.17b}$$

6. CONCLUSIONS

A method is developed and demonstrated for investigating the nonlinear and buckling analysis of planar curved and circular compressive bars. The more important results of this procedure can be concluded as follows:

1. The possibility of decoupling of the nonlinear system (in terms of generalized forces and displacements) governing the equilibrium of the bar, in which the effect of transverse shear has been taken into account;
2. Based on hypergeometric functions, the solution of the homogeneous differential equation (with variable coefficients) governing the radial displacement of a circular thin bar;
3. The proof that, for scallow and flexible circular bars, the previous solution converges to an exact one corresponding to the case of constant axial force acting along the central line of the bar;
4. The determination of the critical buckling load for several cases of shallow or deep flexible circular bars; and
5. The conclusion that analogous results with those of Euler's buckling theory for straight rods are valid.

As a final remark, it is worthwhile noting that from the results included in Tables 1–5 only those of Table 1 have been already derived by Timoshenko (Ref.[1]); a fact that partially confirms the validity of the method.

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